

Dynamic bifurcation and instability of Dean problem

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Abstract

The main objective of this paper is to address the instability and dynamical bifurcation of the Dean problem. A nonlinear theory is obtained for the Dean problem, leading in particular to rigorous justifications of the linear theory used by physicists, and the vortex structure. The main technical tools are the dynamic bifurcation theory [15] developed recently by Ma and Wang.

keywords

Dean problem; instability; dynamic bifurcation.

1 Introduction

The instability of rotating flows is an important issue in fluid dynamics. The problem go back to the pioneering work of Rayleigh [16]. He also considered a basic swirling flow of an inviscid fluid which moves with angular velocity $\Omega(r)$, an arbitrary function of the distance r from the axis of rotation, which

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is determined by the radial rate of change of pressure, then Rayleigh's criterion for stability is "the square of the circulation, $r^2\Omega^2$, is an increasing function of r ." Synge has pointed out that Rayleigh's argument is inconclusive, and that a rigorous demonstration of stability can only be determined by considering the stability of the basic flow to arbitrary perturbations. Further, Synge showed that Rayleigh's criterion is valid for arbitrary perturbations which are symmetric with respect to the axis of rotation [2]. However, Rayleigh's criterion can predicts instability for an inviscid rotating fluid. The stability of an inviscid fluid with respect to non-axisymmetric disturbance has also been studied by Bisshopp [4] and Krueger and DiPrima [9] in the narrow-gap approximation.

Note that the instability of viscous rotation flows also have been considered by several authors including Taylor [17] and Dean [4]. In 1923, Taylor [17] conducted a famous experiment, and observed and studied the stability of an incompressible viscous flow between two rotating coaxial cylinders. He found that when the Taylor number T is smaller than a critical value $T_c > 0$, called the critical Taylor number, the basic flow, called the Couette flow, is stable. While the Taylor number crosses the critical number, the Couette flow breaks into a cellular pattern that is radially symmetric. Such instability called Taylor instability. More research about Taylor instability see [1, 2, 5, 8, 19]. In 1928, Dean [4] found that a similar type of instability can also occur when a viscous fluid in a curved channel owing to a pressure gradient acting round the channel. Such instability is called Dean instability, which has been considered again by Hämmerl [7] and Walowit, Tsao and DiPrima [18]. For Couette flow, especially, in the case that the cylinders rotate in the same direction, a simple formula for predicting the critical speed is derived in [18]. The effect of a radial temperature gradient on the stability of Couette flow is also considered.

Recently, Ma and Wang [10, 15] have developed a bifurcation theory [9] for nonlinear partial differential equations, which has been used to develop a nonlinear analysis for the Rayleigh-Bénard convections [11, 12] and Taylor stability and bifurcation [13, 14]. This bifurcation theory is centered on a new notion of bifurcation, called attractor bifurcation, for nonlinear evolution equations. In particular, based on this bifurcation theory, for the Rayleigh-Bénard convection problem, Ma and Wang have shown [11, 12] that the problem bifurcates from the trivial solution to an attractor \mathcal{A}_R when the Rayleigh number R crosses the first critical Rayleigh number R_c , for all physically sound boundary conditions and regardless of the multiplicity of the eigenvalue R_c for the linear problem. It is worth pointing out that the instability of viscous flows is the nonlinear phenomena.

Motivated by the above papers [11–14], our purpose of this article is to address the dynamic transition of the Dean problem under narrow gap approximation. The main technical tool are the dynamical bifurcation the-

ory [10, 15]. The basis of our research is a basic solution of Dean problem as follows

$$u_\theta^0 = \frac{1}{2\rho\nu} \left(\frac{\partial p}{\partial \theta} \right)_0 \left(r \ln r + Ar + \frac{B}{r} \right),$$

where A and B are determined by boundary velocity. In this paper, we shall get the expressions of A and B by the physical assumption, which are different from expressions in [4, 7, 18] where the boundary velocity vanishes. Based on the basic solution determined by boundary condition and narrow gap approximation, we get a new simplified governing model for the Dean problem. In addition, the expressions of A and B guarantee that the linear part of dynamical equation is symmetrical.

The article is organized as follows. In Section 2, we will give one type of boundary conditions for the steady state solution to guarantee that the corresponding linear part is symmetrical. The simplified model and the set-up are given in Section 2, and the principle of exchange of stability(PES) is given in Section 3, and the main theorems are proved in Section 4.

2 Simplified Governing equations

2.1 the basic flow

The governing equations of a viscous incompressible fluid in the space between two concentric cylinders are the following equations in the cylindrical coordinates, which are given by

$$\begin{cases} \frac{\partial u_r}{\partial t} + (u \cdot \nabla)u_r - \frac{u_\theta^2}{r} = \nu(\Delta u_r - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta}) - \frac{1}{\rho} \frac{\partial p}{\partial r}, \\ \frac{\partial u_\theta}{\partial t} + (u \cdot \nabla)u_\theta + \frac{u_r u_\theta}{r} = \nu(\Delta u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta}) - \frac{1}{r\rho} \frac{\partial p}{\partial \theta}, \\ \frac{\partial u_z}{\partial t} + (u \cdot \nabla)u_z = \nu \Delta u_z - \frac{1}{\rho} \frac{\partial p}{\partial z}, \\ \frac{\partial(ru_r)}{\partial r} + \frac{\partial u_\theta}{\partial \theta} + \frac{\partial(ru_z)}{\partial z} = 0, \end{cases} \quad (2.1)$$

where ν is the kinematic viscosity, ρ is the density, $u = (u_r, u_\theta, u_z)$ is the velocity field, p is the pressure function, and

$$\begin{aligned} (u \cdot \nabla) &= u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r} \frac{\partial}{\partial \theta} + u_z \frac{\partial}{\partial z}, \\ \Delta &= \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}. \end{aligned}$$

To simplify (2.1), we make the following physical assumptions:

1. narrow gap approximation,

$$R_2 - R_1 \leq \frac{R_1 + R_2}{2}, \quad (2.2)$$

where R_1 and R_2 are the radius of of the two concentric cylinders.

2. the steady state solution of (2.1) only related to the θ -direction and only depend on the variable r , i.e.,

$$u = (0, \bar{u}_\theta(r), 0). \quad (2.3)$$

3. the partial derivatives of pressure function \tilde{p} with respect to θ is a constant, i.e.,

$$\frac{\partial \bar{p}}{\partial \theta} = \left(\frac{\partial p}{\partial \theta} \right)_0 = \text{const.} \quad (2.4)$$

From [5], the basic flow for (2.1) is a steady state solution, defined by

$$\begin{cases} u_r = u_z = 0, & u_\theta = \bar{u}_\theta(r) = \frac{1}{2\rho\nu} \left(\frac{\partial p}{\partial \theta} \right)_0 \left(r \ln r + Ar + \frac{B}{r} \right), \\ \frac{1}{\rho} \frac{\partial \bar{p}}{\partial r} = \frac{u_\theta^2(r)}{r}, & \frac{\partial \bar{p}}{\partial \theta} = \left(\frac{\partial p}{\partial \theta} \right)_0. \end{cases} \quad (2.5)$$

Remark 1. In this paper, the basic flow (2.4) we considered is different from the Couette flow for the Taylor problem.

By the boundary conditions

$$\bar{u}_\theta(R_1) = \frac{1}{2\rho\nu} \left(\frac{\partial p}{\partial \theta} \right)_0 R_2^2, \quad \bar{u}_\theta(R_2) = \frac{1}{2\rho\nu} \left(\frac{\partial p}{\partial \theta} \right)_0 R_1^2. \quad (2.6)$$

We also assume that the pressure p is a function depending on θ and r , satisfying that

$$\frac{1}{\rho} \frac{\partial p}{\partial r} = \frac{u_\theta^2(r)}{r}, \quad \frac{\partial p}{\partial \theta} = \left(\frac{\partial p}{\partial \theta} \right)_0. \quad (2.7)$$

Basing on the above assumptions, together (2.5) with (2.6), we get

$$A = - \frac{R_2^2 \ln R_2 - R_1^2 \ln R_1 + R_2^2 R_1 - R_1^2 R_2}{R_2^2 - R_1^2}, \quad (2.8)$$

$$B = \frac{R_1^2 R_2^2 (\ln R_2 - \ln R_1) + R_2^4 R_1 - R_1^4 R_2}{R_2^2 - R_1^2}. \quad (2.9)$$

Remark 2. Note that (2.8), (2.9) and the narrow gap approximation assumption (2.2) can guarantee the linear part of the perturbation equation by (2.5) is symmetrical.

2.2 perturbed non-dimensionless equation

In the following, in order to investigate the stability of the flow described by (2.5), we need to consider the perturbed state

$$u_r, \quad \bar{u}_\theta + u_\theta, \quad u_z, \quad p + \bar{p}.$$

Assume that the perturbations are axi-symmetric and independent of θ , we derive from (2.1) that

$$\left\{ \begin{array}{l} \frac{\partial u_r}{\partial t} + (u \cdot \nabla)u_r - \frac{u_\theta^2}{r} = \nu \left(\Delta u_r - \frac{u_r}{r^2} \right) - \frac{1}{\rho} \frac{\partial p}{\partial r} \\ \quad + \frac{1}{2\nu\rho} \left(\frac{\partial p}{\partial \theta} \right)_0 \left(2 \ln r + 2A + \frac{2B}{r^2} \right) u_\theta, \\ \frac{\partial u_\theta}{\partial t} + (u \cdot \nabla)u_\theta + \frac{u_r u_\theta}{r} = \nu \left(\Delta u_\theta - \frac{u_\theta}{r^2} + \frac{1}{\nu\rho} \frac{1}{r} \right) \\ \quad - \frac{1}{2\nu\rho} \left(\frac{\partial p}{\partial \theta} \right)_0 (2 \ln r + 2A + 1) u_r, \\ \frac{\partial u_z}{\partial t} + (u \cdot \nabla)u_z = \nu \Delta u_z - \frac{1}{\rho} \frac{\partial p}{\partial z}, \\ \frac{\partial}{\partial r}(ru_r) + \frac{\partial}{\partial z}(ru_z) = 0, \end{array} \right. \quad (2.10)$$

where

$$(u \cdot \nabla) = u_r \frac{\partial}{\partial r} + u_z \frac{\partial}{\partial z}, \quad (2.11)$$

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \quad (2.12)$$

To derive the non-dimensionless form of equations (2.10), let

$$\begin{aligned} (x, t) &= \left(lx', \frac{l^2}{\nu} t' \right), & (x &= (r, r\theta, z)), \\ (u, p) &= \left(\frac{\nu}{l} u', \frac{\rho \nu^2}{l^2} p' \right), & (u &= (u_r, u_\theta, u_z)). \end{aligned}$$

Omitting the primes, we obtain the non-dimensionless form of (2.10) as follows

$$\left\{ \begin{array}{l} \frac{\partial u_r}{\partial t} + (u \cdot \nabla)u_r - \frac{u_\theta^2}{r} = \Delta u_r - \frac{u_r}{r^2} - \frac{\partial p}{\partial r} + \left(\frac{\partial p}{\partial \theta} \right)_0 \left(\ln r + A + \frac{B}{r^2} \right) u_\theta, \\ \frac{\partial u_\theta}{\partial t} + (u \cdot \nabla)u_\theta + \frac{u_r u_\theta}{r} = \Delta u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r} \left(\frac{\partial p}{\partial \theta} \right)_0 \\ \quad - \left(\frac{\partial p}{\partial \theta} \right)_0 \left(\ln r + A + \frac{1}{2} \right) u_r, \\ \frac{\partial u_z}{\partial t} + (u \cdot \nabla)u_z = \Delta u_z - \frac{\partial p}{\partial z}, \\ \frac{\partial}{\partial r}(ru_r) + \frac{\partial}{\partial z}(ru_z) = 0, \end{array} \right. \quad (2.13)$$

Taking the length scale $l = R_2 - R_1$, then the narrow gap condition is given by

$$\frac{l}{R_1} \ll 1. \quad (2.14)$$

Furthermore, by (2.14) and (2.8), we have

$$\begin{aligned} & \ln r + A + \frac{1}{2} \\ &= \ln r - \frac{R_2^2 \ln R_2 - R_1^2 \ln R_1 + R_2^2 R_1 - R_1^2 R_2}{R_2^2 - R_1^2} + \frac{1}{2} \\ &= \ln r - \ln R_2 - \frac{R_1^2 (\ln R_2 - \ln R_1)}{R_2^2 - R_1^2} - \frac{R_1 R_2}{R_1 + R_2} + \frac{1}{2} \\ &= \ln \frac{r}{R_2} - \frac{\ln(1 + l/R_1)}{(1 + l/R_1)^2 - 1} - \frac{R_1 R_2}{R_1 + R_2} + \frac{1}{2} \\ &\sim -\frac{l/R_1}{l/R_1(1 + l/R_1)} - \frac{\ln(1 + l/R_1)}{(1 + l/R_1)^2 - 1} - \frac{R_1 R_2}{R_1 + R_2} + \frac{1}{2} \\ &\sim -\frac{R_1 R_2}{R_1 + R_2}, \end{aligned} \quad (2.15)$$

Moreover,

$$\begin{aligned}
& \ln r + A + \frac{B}{r^2} \\
&= \ln r + A + \frac{1}{2} + \frac{B}{r^2} - \frac{1}{2} \\
&\sim -\frac{R_1 R_2}{R_1 + R_2} + \frac{1}{r^2} \frac{R_1^2 R_2^2 (\ln R_2 - \ln R_1) + R_2^4 R_1 - R_1^4 R_2}{R_2^2 - R_1^2} - \frac{1}{2} \\
&= -\frac{R_1 R_2}{R_1 + R_2} + \frac{R_2^2}{r^2} \frac{R_1^2 (\ln R_2 - \ln R_1)}{R_2^2 - R_1^2} + \frac{R_1 R_2}{r^2} \frac{R_2^3 - R_1^3}{R_2^2 - R_1^2} - \frac{1}{2} \\
&\sim -\frac{R_1 R_2}{R_1 + R_2} + \frac{1}{2} + \frac{R_2^2 + R_1 R_2 + R_1^2}{R_2^2 - R_1^2} - \frac{1}{2} \\
&\sim \frac{2R_1 R_2}{R_1 + R_2}.
\end{aligned} \tag{2.16}$$

Under the assumption (2.14), we can neglect the terms containing r^{-n} ($n \geq 1$) in (2.13). Furthermore, by (2.15) and (2.16), we have

$$\begin{cases}
\frac{\partial u_r}{\partial t} + (u \cdot \nabla) u_r = \Delta u_r - \frac{\partial p}{\partial r} + \left(\frac{\partial p}{\partial \theta} \right)_0 \frac{2R_1 R_2}{R_1 + R_2} u_\theta \\
\frac{\partial u_\theta}{\partial t} + (u \cdot \nabla) u_\theta = \Delta u_\theta + \left(\frac{\partial p}{\partial \theta} \right)_0 \frac{R_1 R_2}{R_1 + R_2} u_r, \\
\frac{\partial u_z}{\partial t} + (u \cdot \nabla) u_z = \Delta u_z - \frac{\partial p}{\partial z}, \\
\frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z} = 0,
\end{cases} \tag{2.17}$$

where Δ is defined as follows

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2}.$$

Let $u_r = u'_r$, $u_\theta = \frac{1}{\sqrt{2}} u'_\theta$, $u_z = u'_z$, $p = p'$. Omitting the primes, we can rewrite the equations (2.17) as follows

$$\begin{cases}
\frac{\partial u_r}{\partial t} = \Delta u_r - \frac{\partial p}{\partial r} + \left(\frac{\partial p}{\partial \theta} \right)_0 \frac{\sqrt{2} R_1 R_2}{R_1 + R_2} u_\theta - (u \cdot \nabla) u_r \\
\frac{\partial u_\theta}{\partial t} = \Delta u_\theta + \left(\frac{\partial p}{\partial \theta} \right)_0 \frac{\sqrt{2} R_1 R_2}{R_1 + R_2} u_r - (u \cdot \nabla) u_\theta \\
\frac{\partial u_z}{\partial t} = \Delta u_z - \frac{\partial p}{\partial z} - (u \cdot \nabla) u_z \\
\frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z} = 0,
\end{cases} \tag{2.18}$$

Remark 3. It should be pointed out that the existence of global weak solution, the regularity of global weak solution and the existence of global attractor for the simplified model (2.18) under various boundary conditions has been obtained in our other submitted papers.

2.3 Abstract operator form

For convenience, we denote $\lambda = \left(\frac{\partial p}{\partial \theta}\right)_0 \frac{\sqrt{2}R_1R_2}{R_1+R_2}$. In this paper, we only consider the following free boundary conditions:

$$\begin{cases} u_r = 0, u_\theta = 0, \frac{\partial u_z}{\partial r} = 0, & \text{when } r = r_1, r = r_2, \\ \frac{\partial u_r}{\partial r} = \frac{\partial u_\theta}{\partial z} = 0, u_z = 0, & \text{when } z = 0, L. \end{cases} \quad (2.19)$$

where $r_1 = \frac{R_1}{l}$ and $r_2 = \frac{R_2}{l}$ are non-dimensionless constants.

Let

$$\begin{aligned} H &= \left\{ u = ((u_r, u_z), u_\theta) \in H^2(\Omega, \mathbb{R}^2) \times H^2(\Omega) \mid u \text{ satisfies (2.19)} \right. \\ &\quad \left. \text{and } \frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z} = 0 \right\}, \\ H_1 &= \left\{ u = ((u_r, u_z), u_\theta) \in L^2(\Omega, \mathbb{R}^2) \times L^2(\Omega) \mid u \text{ satisfies (2.19)} \right. \\ &\quad \left. \text{and } \frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z} = 0 \right\}. \end{aligned} \quad (2.20)$$

Define the mappings $A, B, G : H \rightarrow H_1$ as follows

$$\begin{aligned} Au &= -(P(\Delta u_r, \Delta u_z), \Delta u_\theta), \\ Bu &= (P(u_\theta, 0), u_r), \\ Gu &= (P((u \cdot \nabla)u_r, (u \cdot \nabla)u_z), (u \cdot \nabla)u_\theta), \end{aligned}$$

where P is the Leray projection.

Setting $L_\lambda = -A + \lambda B$, then the equation (2.18) is equivalent to the following operator equation

$$\begin{cases} \frac{du}{dt} = L_\lambda u + Gu, & t > 0 \\ u_0 = (u_{r0}, u_{z0}, u_{\theta0}) \end{cases} \quad (2.21)$$

where u_0 is the initial value of (2.18).

3 Principle of exchange of stability

Let the eigenvalue of operator L_λ and the corresponding eigenvector be β and $v = (v_r, v_\theta, v_z)$ respectively such that

$$L_\lambda v = \beta v. \quad (3.1)$$

Obviously, (3.1) is equivalent to

$$\begin{cases} \Delta v_r + \lambda v_\theta - \frac{\partial p}{\partial r} = \beta v_r, \\ \Delta v_\theta + \lambda v_r = \beta v_\theta, \\ \Delta v_z - \frac{\partial p}{\partial z} = \beta v_z, \\ \frac{\partial v_r}{\partial r} + \frac{\partial v_z}{\partial z} = 0. \end{cases} \quad (3.2)$$

Clearly, $v_r, v_\theta \in W_1$ and $v_z \in W_2$ respectively, where

$$W_1 = \left\{ \psi \in H^2(\Omega) \mid \psi = 0, \text{ when } r = r_1, r_2; \frac{\partial \psi}{\partial z} = 0, \text{ when } z = 0, L \right\},$$

$$W_2 = \left\{ \psi \in H^2(\Omega) \mid \frac{\partial \psi}{\partial r} = 0, \text{ when } r = r_1, r_2; \psi = 0, \text{ when } z = 0, L \right\}.$$

We know that

$$\Delta : W_1 \rightarrow S_1$$

and

$$\Delta : W_2 \rightarrow S_2$$

are isomorphic, where

$$S_1 = \left\{ \psi \in L^2(\Omega) \mid \psi = 0, \text{ when } r = r_1, r_2; \frac{\partial \psi}{\partial z} = 0, \text{ when } z = 0, L \right\},$$

$$S_2 = \left\{ \psi \in L^2(\Omega) \mid \frac{\partial \psi}{\partial r} = 0, \text{ when } r = r_1, r_2; \psi = 0, \text{ when } z = 0, L \right\}.$$

Noticing that the eigenvectors e_{mn} and \bar{e}_{mn} of Δ in W_1 and W_2 are the orthogonal basis of W_1 and W_2 respectively. The eigenvalues λ_{mn} and eigenvectors e_{mn} of Δ in W_1 are as follows

$$\lambda_{mn} = -\pi^2 \left(\frac{m^2}{L^2} + n^2 \right), \quad e_{mn} = \cos \frac{m\pi z}{L} \sin n\pi(r - r_1).$$

The eigenvalues $\bar{\lambda}_{mn}$ and eigenvectors \bar{e}_{mn} of Δ in W_2 are as follows

$$\bar{\lambda}_{mn} = \lambda_{mn}, \quad \bar{e}_{mn} = \sin \frac{m\pi z}{L} \cos n\pi(r - r_1),$$

where $m, n = 1, 2, \dots$.

Hence, v_r, v_θ are linear combinations of the eigenvectors e_{mn} and v_z is linear combination of the eigenvectors \bar{e}_{mn} , that is

$$\begin{aligned} v_r &= \sum_{m,n=1}^{\infty} a_r^{mn} e_{mn}, \\ v_\theta &= \sum_{m,n=1}^{\infty} a_\theta^{mn} e_{mn}, \\ v_z &= \sum_{m,n=1}^{\infty} a_z^{mn} \bar{e}_{mn}. \end{aligned} \tag{3.3}$$

Substituting (3.3) into (3.2) to get

$$\left\{ \begin{aligned} \sum_{m,n=1}^{\infty} (a_r^{mn} \lambda_{mn} e_{mn} + \lambda a_\theta^{mn} e_{mn}) - \frac{\partial p}{\partial r} &= \beta \sum_{m,n=1}^{\infty} a_r^{mn} e_{mn}, \\ \sum_{m,n=1}^{\infty} (a_\theta^{mn} \lambda_{mn} e_{mn} + \lambda a_r^{mn} e_{mn}) &= \beta \sum_{m,n=1}^{\infty} a_\theta^{mn} e_{mn}, \\ \sum_{m,n=1}^{\infty} a_z^{mn} \lambda_{mn} \bar{e}_{mn} - \frac{\partial p}{\partial z} &= \beta \sum_{m,n=1}^{\infty} a_z^{mn} e_{mn}, \\ \sum_{m,n=1}^{\infty} \left(n\pi a_r^{mn} \cos \frac{m\pi z}{L} \cos n\pi(r-r_1) + \frac{m\pi}{L} a_z^{mn} \cos \frac{m\pi z}{L} \cos n\pi(r-r_1) \right) &= 0. \end{aligned} \right. \tag{3.4}$$

Eliminating p of (3.4) to get

$$\left\{ \begin{aligned} & - \sum_{m,n=1}^{\infty} (a_r^{mn} \lambda_{mn} + \lambda a_\theta^{mn}) \frac{m\pi}{L} \sin \frac{m\pi z}{L} \sin n\pi(r-r_1) \\ & \quad + \sum_{m,n=1}^{\infty} a_z^{mn} \lambda_{mn} n\pi \sin \frac{m\pi z}{L} \sin n\pi(r-r_1) \\ & \quad = -\beta \sum_{m,n=1}^{\infty} \left(a_r^{mn} \frac{m\pi}{L} - n\pi a_z^{mn} \right) \sin \frac{m\pi}{L} z \sin n\pi(r-r_1), \\ & \sum_{m,n=1}^{\infty} (a_\theta^{mn} \lambda_{mn} e_{mn} + \lambda a_r^{mn} e_{mn}) = \beta \sum_{m,n=1}^{\infty} a_\theta^{mn} e_{mn}, \\ & \sum_{m,n=1}^{\infty} \left(n\pi a_r^{mn} \cos \frac{m\pi z}{L} \cos n\pi(r-r_1) + \frac{m\pi}{L} a_z^{mn} \cos \frac{m\pi z}{L} \cos n\pi(r-r_1) \right) = 0. \end{aligned} \right. \tag{3.5}$$

Furthermore, we obtain

$$\begin{cases} -\frac{m\pi}{L}(a_r^{mn}\lambda_{mn} + \lambda a_\theta^{mn}) + n\pi a_z^{mn}\lambda_{mn} = -\beta(a_r^{mn}\frac{m\pi}{L} - a_z^{mn}n\pi), \\ a_\theta^{mn}\lambda_{mn} + \lambda a_r^{mn} = \beta a_\theta^{mn}, \\ na_r^{mn} + \frac{m}{L}a_z^{mn} = 0, \end{cases} \quad (3.6)$$

Then, we can deduce from (3.6) that

$$\begin{pmatrix} \lambda_{mn} & \frac{\lambda}{1+\frac{n^2L^2}{m^2}} \\ \lambda & \lambda_{mn} \end{pmatrix} \begin{pmatrix} a_r^{mn} \\ a_\theta^{mn} \end{pmatrix} = \beta \begin{pmatrix} a_r^{mn} \\ a_\theta^{mn} \end{pmatrix}. \quad (3.7)$$

Let

$$A_{mn} = \begin{pmatrix} \lambda_{mn} & \frac{\lambda}{1+\frac{n^2L^2}{m^2}} \\ \lambda & \lambda_{mn} \end{pmatrix}, \quad X_{mn} = \begin{pmatrix} a_r^{mn} \\ a_\theta^{mn} \end{pmatrix},$$

we rewrite (3.7) as follows

$$A_{mn}X_{mn} = \beta X_{mn}. \quad (3.8)$$

Obviously, the eigenvalue of (3.2) is also the eigenvalue of (3.8). It follows from (3.7) that

$$\beta_{mn}^1 = \lambda_{mn} + \frac{\lambda}{\sqrt{1 + \frac{n^2L^2}{m^2}}}, \quad (3.9)$$

$$\beta_{mn}^2 = \lambda_{mn} - \frac{\lambda}{\sqrt{1 + \frac{n^2L^2}{m^2}}}. \quad (3.10)$$

and the eigenvectors corresponding $\beta_{mn}^i (i = 1, 2)$ are given by

$$f_{mn}^i = \frac{2}{L} \left((-1)^{i+1} \frac{1}{\sqrt{1 + \frac{n^2L^2}{m^2}}} e_{mn}, e_{mn}, (-1)^i \frac{nL}{m} \frac{1}{\sqrt{1 + \frac{n^2L^2}{m^2}}} \bar{e}_{mn} \right). \quad (3.11)$$

From (3.11), it is easy to get the following properties of f_{mn}^i .

Proposition 1. *Let f_{mn}^i satisfy (3.11), then*

1. $\{f_{mn}^i\}$ is an orthogonal basis of H and H_1 , and

$$\begin{cases} \langle f_{mn}^i, f_{sk}^j \rangle = 0, & (i, m, n) \neq (j, s, k), \\ \langle f_{mn}^i, f_{sk}^j \rangle = 1, & (i, m, n) = (j, s, k), \end{cases}$$

- 2.

$$\langle G(f_{mn}^i, f_{sk}^j), f_{lh}^p \rangle = -\langle G(f_{mn}^i, f_{lh}^p), f_{sk}^j \rangle,$$

where $G(f_{mn}^i, f_{sk}^j) = (f_{mn}^i \cdot \nabla) f_{sk}^j$,

- 3.

$$\langle G(f_{mn}^i, f_{sk}^j), f_{sk}^j \rangle = 0.$$

3.1 First eigenvalue and critical number

Obviously, it is easy to check that

$$\beta_{mn}^2 = \lambda_{mn} - \frac{\lambda}{\sqrt{1 + \frac{n^2 L^2}{m^2}}} < 0, \quad \text{for } \lambda > 0.$$

Hence, the first eigenvalue of L_λ is determined by the following formulation:

$$\beta_1 = \max_{m,n \in N} \beta_{mn}^1 = \max_{m \in N} \left(-\pi^2 \left(1 + \frac{m^2}{L^2}\right) + \frac{\lambda}{\sqrt{1 + \frac{L^2}{m^2}}} \right). \quad (3.12)$$

(3.12) implies

$$\beta_{m1}^1 = 0 \quad \text{if } \lambda = \lambda_m = \pi^2 \sqrt{1 + \frac{L^2}{m^2}} \left(1 + \frac{m^2}{L^2}\right). \quad (3.13)$$

Note that λ_m is the function of m , which can obtain the minimum λ_0 when $m = m_0 \in N$.

Let $\lambda_0 = \pi^2 \sqrt{1 + \frac{L^2}{m_0^2}} \left(1 + \frac{m_0^2}{L^2}\right)$. Then there exists a positive $\delta > 0$ such that $\lambda_0 - \delta < \lambda < \lambda_0 + \delta$, and $\beta_1 = \beta_{m_0 1}^1$,

$$\begin{cases} \beta_{m_0 1}^1 \begin{cases} < 0, & \lambda_0 - \delta < \lambda < \lambda_0, \\ = 0, & \lambda = \lambda_0, \\ > 0, & \lambda_0 < \lambda < \lambda_0 + \delta, \end{cases} \\ \beta_{mn}^i < 0, \quad (i, m, n) \neq (1, m_0, 1). \end{cases} \quad (3.14)$$

Let

$$E_1 = \bigcup_{k \in N} \{x \in H \mid (L_{\lambda_0} - \beta_{m_0 1}^1(\lambda_0))^k x = 0\},$$

which means that $\dim E_1 = 1$.

4 Main conclusions

Based on Theorem 2.3.1 [15], we can get the following conclusions.

Theorem 4.1. *For system (2.21), the following conclusion hold.*

1. *if $0 \leq \lambda < \lambda_0$, it does not have bifurcation, and the only equilibrium point $u = 0$ is globally asymptotically stable;*
2. *there exists a constant $\varepsilon > 0$ such that $\lambda_0 < \lambda < \lambda_0 + \varepsilon$, system (2.21) bifurcates from $(0, \lambda_0)$ to an attractor Ω_λ , and attractor Ω_λ attracts H_1/Γ , where Γ is the stable manifold $(0, 0)$, which has codimension 1 in H_1 .*

Proof. It is obvious that the eigenvalue $\beta_{mn}^i < 0$ when $0 < \lambda < \lambda_0$. Hence, the conclusion 1 naturally holds.

Note that for any $u \in H$,

$$\langle G(u), u \rangle = \int_{\Omega} (u \cdot u) u dx = -\frac{1}{2} \left(\frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z} u^2 \right) dx = 0.$$

Based on the results in [6], system (2.21) has a global attractor in H , that is, all invariant sets are uniformly bounded. Then we only need to prove that 0 is the unique invariant set as $\lambda = \lambda_0$. We infer from (3.14) that any invariant set Σ of system (2.21) is only in E_1 . Suppose Σ has a non-zero element, for $u = (u_r, u_{\theta}, u_z) \in \Sigma$, we know u satisfies the following equations

$$\begin{cases} \frac{\partial u_r}{\partial t} = -\frac{\partial p}{\partial r} - (u \cdot \nabla) u_r \\ \frac{\partial u_{\theta}}{\partial t} = -(u \cdot \nabla) u_{\theta} \\ \frac{\partial u_z}{\partial t} = -\frac{\partial p}{\partial z} - (u \cdot \nabla) u_z \\ \frac{\partial}{\partial r}(u_r) + \frac{\partial}{\partial z}(u_z) = 0, \end{cases}$$

Clearly, $bu(bt)$ satisfies the above equations as well, $bu(bt) \in b\Sigma \subset E_1$, where b is any real number. This contradicts to the boundaries of Σ . Then the invariant set of the system (2.21) can only be 0, and 0 is a globally asymptotically stable attractor. \square

Theorem 4.2. *The set Ω_{λ} given in theorem 4.1, which contains two equilibrium point, that is $\Omega_{\lambda} = (\Omega_1^{\lambda}, \Omega_2^{\lambda})$. And there are following conclusion:*

$$1. \quad \Omega_1^{\lambda} = a\sqrt{\beta_{m_0 1}^1} f_{m_0 1}^1 + o(\sqrt{\beta_{m_0 1}^1}), \quad \Omega_2^{\lambda} = -a\sqrt{\beta_{m_0 1}^1} f_{m_0 1}^1 + o(\sqrt{\beta_{m_0 1}^1}),$$

where a is a positive real number;

2. H_1 is decomposed into two open sets U_1 and U_2 :

$$H_1 = \overline{U_1} + \overline{U_2}, \quad U_1 \cap U_2 = \phi, \quad 0 \in \partial U_1 \cap \partial U_2,$$

such that $\Omega_i^{\lambda} \in U_i (i=1,2)$, and

$$\lim_{t \rightarrow \infty} \|u(t, \varphi) - \Omega_i^{\lambda}\| = 0, \quad \text{for any } \varphi \in U_i.$$

Proof. We only need to prove the conclusion 1.

It is clear that the steady state equation of (2.21) is

$$L_{\lambda} u + Gu = 0, \quad u \in H. \quad (4.1)$$

Let the first feature space of L_λ be E , and the orthogonal complement of E be E_2 , then

$$H = E_1 \oplus E_2, \quad u = u_1 + u_2, \quad u_i \in E_i, i = 1, 2.$$

Then (4.1) can be rewritten as

$$L_\lambda u_1 + P_1 G(u_1 + u_2) = 0, \quad (4.2)$$

$$L_\lambda u_2 + P_2 G(u_1 + u_2) = 0. \quad (4.3)$$

where $P_1 : H \rightarrow E_1$ and $P_2 : H \rightarrow E_2$ are projections.

Thus, (4.2)-(4.3) are equivalent to the equations

$$\langle L_\lambda u_1, f_{m_0 1}^1 \rangle + \langle G(u_1 + u_2), f_{m_0 1}^1 \rangle = 0, \quad (4.4)$$

$$\langle L_\lambda u_2, f_{mn}^j \rangle + \langle G(u_1 + u_2), f_{mn}^j \rangle = 0, \quad (j, m, n) \neq (1, m_0, 1), \quad (4.5)$$

where $f_{m_0 1}^1$ is the first eigenvector of L_λ . (4.4) is the finite-dimensional bifurcation equations.

For any $u \in H$, u can be expressed as

$$u = u_1 + u_2 = y_{m_0 1}^1 f_{m_0 1}^1 + \sum_{m>1, n, i \geq 1} y_{mn}^i f_{mn}^i, \quad y_{m_0 1}^1, y_{mn}^i \in \mathbb{R}^1.$$

Then (4.4) and (4.5) are equivalent to the following two equations:

$$\begin{aligned} \beta_{m_0 1}^1 y_{m_0 1}^1 + \langle G(u_1 + u_2), f_{m_0 1}^1 \rangle &= 0, \\ \beta_{mn}^i y_{mn}^i + \langle G(u_1 + u_2), f_{mn}^i \rangle &= 0, \quad (i, m, n) \neq (1, m_0, 1), \end{aligned}$$

By the proposition 1 and the implicit function theorem [10], we can get

$$\beta_{m_0 1}^1 y_{m_0 1}^1 - \sum_{m>1, n, i \geq 1} y_{m_0 1}^1 y_{mn}^i \langle G(f_{m_0 1}^1), f_{mn}^i \rangle + O(|y_{mn}^i|^2) = 0, \quad (4.6)$$

$$y_{mn}^i = -\frac{1}{\beta_{mn}^i} (y_{m_0 1}^1)^2 \langle G(f_{m_0 1}^1), f_{mn}^i \rangle + o(|y_{m_0 1}^1|^2). \quad (4.7)$$

Taking (4.7) into (4.6), we have

$$\beta_{m_0 1}^1 y_{m_0 1}^1 - 2\gamma (y_{m_0 1}^1)^3 + o(|y_{m_0 1}^1|^3) = 0, \quad (4.8)$$

where

$$\gamma = \sum_{m>1, n, i \geq 1} -\frac{1}{\beta_{mn}^i} \langle G(f_{m_0 1}^1), f_{mn}^i \rangle^2 > 0.$$

From (3.14), we can infer that (4.8) bifurcates two singular points from $(0, \lambda_0)$ when $\lambda > \lambda_0$:

$$(y_{m_0 1}^1)_{1,2} = \pm \sqrt{\frac{\beta_{m_0 1}^1}{\gamma}} + o(|\beta_{m_0 1}^1|^{\frac{1}{2}}).$$

Then

$$\Omega_1^\lambda = a\sqrt{\beta_{m_0 1}^1}f_{m_0 1}^1 + o(|\beta_{m_0 1}^1|^{\frac{1}{2}}), \quad \Omega_2^\lambda = -a\sqrt{\beta_{m_0 1}^1}f_{m_0 1}^1 + o(|\beta_{m_0 1}^1|^{\frac{1}{2}}), \quad (4.9)$$

where $a = \frac{1}{\sqrt{\gamma}} > 0$. □

Physical explanation of theorem 4.2. The fact is that R_1, R_2, ρ and ν are fixed, and we can only adjust the value of $\left(\frac{\partial p}{\partial \theta}\right)_0$. Then (3.14) and the expression of λ imply that the basic flow becomes unstable, and the movement of the flow in r and z directions will occur if $\left(\frac{\partial p}{\partial \theta}\right)_0$ exceeds a certain critical value. The structure of flow in r and z directions is shown in Figure 1, and the number of vortexes is determined by L and m_0 .

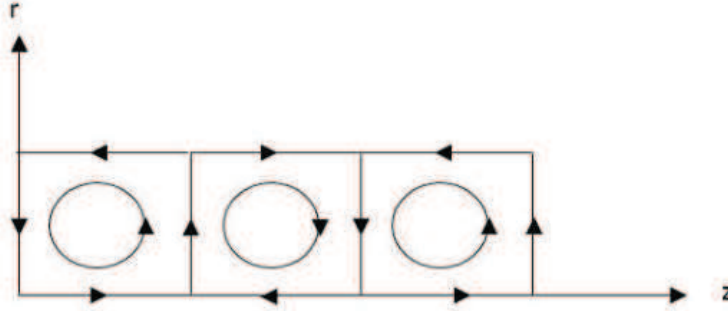


Figure 1: Structure

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